EVERWIN MATRIC. HR. SEC. SCHOOL

STD: XII Material of the month – MARCH

Maths

Theorem 1.1: For every square matrix A of order n, A(adj A)=(adj A)A=|A| I_n

Proof: We prove the theorem for n=3, consider $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

and we know tha $a_{i1}A_{j1}+a_{i2}A_{j2}+\ldots\ldots+a_{in}A_{jn}=\begin{cases} |A| \text{ if } i=j \\ 0 \text{ if } i\neq j \end{cases}$

$$A(\text{adj } A)=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

$$=\begin{bmatrix} a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13} & a_{11}A_{21}+a_{12}A_{22}+a_{13}A_{23} & a_{11}A_{31}+a_{12}A_{32}+a_{13}A_{33} \\ A_{21}A_{11}+a_{22}A_{12}+a_{23}A_{13} & a_{21}A_{21}+a_{22}A_{22}+a_{23}A_{23} & a_{21}A_{31}+a_{22}A_{32}+a_{23}A_{33} \\ A_{31}A_{11}+a_{32}A_{12}+a_{33}A_{13} & a_{31}A_{21}+a_{32}A_{22}+a_{33}A_{23} & a_{31}A_{31}+a_{32}A_{32}+a_{33}A_{33} \end{bmatrix}$$

$$=\begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}=|A|\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$=|A|I \longrightarrow (1)$

$$(\text{adj } A)A=\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$=\begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}=|A|\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$=|A|I \longrightarrow (2)$

Where I_3 is the identity matrix of order 3.

∴ From (1) & (2), A(adj A)=(adj A)A=|A| I_n.

Theorem 1.2: If a square matrix has an inverse, then it is unique.

Proof: Let A be a square matrix of order n and A^{-1} exist.

If possible let there be two inverses B and C for A. Then

$AB=BA=I_n$ and $AC=CA=I_n$

Now $C=C\,I_n=C(AB)=(CA)B=I_nB=B$

∴ C=B

Hence the uniqueness follows.

Theorem 1.3: Let A be a square matrix of order n. Then A^{-1} exist if and only if A is non-singular.

Proof: Suppose A^{-1} exist

To Prove that A is non-singular Since A^{-1} exist, $AA^{-1}=A^{-1}A=I$

Consider, $AA^{-1}=I$

$|AA^{-1}| = |I|$

$|A|\,|A^{-1}|=1\neq 0$

$\Rightarrow |A| \neq 0$

⇒A IS NON-SINGULAR.

Conversely, A is non-singular

To Prove that , A^{-1} exist, since A is non-singular, $\Rightarrow |A| \neq 0$

we know that, A(adj A)=(adj A)A=|A|I ÷by |A|,

$A\left(\frac{1}{|A|}\text{adj } A\right)=\left(\frac{1}{|A|}\text{adj}A\right)A=I$, if $B=\frac{1}{|A|}\text{adj } A$, we get AB=BA=I

$\Rightarrow A^{-1}$ exist and $A^{-1}=B$

$\Rightarrow A^{-1}=\frac{1}{|A|}\text{adj}A$

Theorem 1.4: If A is non-singular then i) $|A^{-1}|=\frac{1}{|A|}$ ii) $(A^T)^{-1}=(A^{-1})^T$ iii) $(\lambda A)^{-1}=\frac{1}{\lambda}A^{-1}$ where λ is a non-zero scalar.

Proof: Given A is non-singular.

$\Rightarrow |A| \neq 0$ and A^{-1} exist.

∴ $AA^{-1}=A^{-1}A=I_n \longrightarrow (1)$

i) Consider $AA^{-1}=I_n$

$|AA^{-1}|=|I_n|$

$|A|\,|A^{-1}| = 1$

$\Rightarrow |A^{-1}| = \frac{1}{|A|}$

ii) From (i) $AA^{-1}=(A^{-1})A=I$

Take transpose, $(AA^{-1})^T=I_n^T$

$(A^{-1})^T A^T = I_n$

$\Rightarrow (A^{-1})^T$ is the inverse of A^T.

$\Rightarrow (A^{-1})^T=(A^T)^{-1}$

iii) multiply and divide by λ, in $AA^{-1}=I$

$(\lambda A).\left(\frac{1}{\lambda}A^{-1}\right)=I$

$\Rightarrow \frac{1}{\lambda}A^{-1}$ is the inverse of λA.

$\Rightarrow \frac{1}{\lambda}A^{-1} = (\lambda A)^{-1}$.

<u>Theorem 1.5</u>: Left cancellation law:

Let A, B and C be square matrices of order n. If A is non-singular and AB=AC then B=C.

Proof: Since A is non-singular and A^{-1} exist. $AA^{-1}=A^{-1}A=I$

Given AB=AC

Premultiply by A^{-1} on both sides,

$A^{-1}(AB)=A^{-1}(AC)$

$(A^{-1}A)B=(AA^{-1})C$ (By associative property)

IB=IC

$\Rightarrow$B=C

<u>Theorem 1.6</u>: Right cancellation law:

Let A, B and C are square matrices of order n. If A is non-singular and BA=CA then B=C.

Proof: A is non-singular and A^{-1} exist.

Post multiply by A^{-1} on both sides in BA=CA.

$(BA)A^{-1} = (CA)A^{-1}$

$B(AA^{-1})=C(AA^{-1})$ (By associative property)

$B\,I_n = C\,I_n$

B = C

<u>Theorem 1.7</u>: (Reversal law of inverses)

If A and B are non-singular matrices of the same order, then the product AB is also non-singular and $(AB)^{-1}=B^{-1}A^{-1}$.

Proof: Given A and B are non-singular.

$\therefore |A| \neq 0$; $|B| \neq 0$ and A^{-1}, B^{-1} exists.

Now $|AB| = |A|\,|B| \neq 0$

$\Rightarrow |AB| \neq 0$

$\Rightarrow$ AB is non-singular and $(AB)^{-1}$ exist.

Now $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$=(AI)A^{-1}$

$=AA^{-1}$

$=I$

Likewise $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$

$=(B^{-1}I)B$

$=B^{-1}B$

$=I$

$\Rightarrow B^{-1}A^{-1}$ is the inverse of AB.

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

<u>Theorem 1.8</u>: Law of double inverse

If A is non-singular and A^{-1} is also non-singular then $(A^{-1})^{-1}=A$

Proof: We know that,

$AA^{-1} = I$

$(AA^{-1})^{-1} = I^{-1}$

$(A^{-1})^{-1}A^{-1} = I$

Post multiply by A on both sides,

$(A^{-1})^{-1}(A^{-1}A) = IA$

$(A^{-1})^{-1}I = A$

$(A^{-1})^{-1} = A$.

EVERWIN MATRIC. HR. SEC. SCHOOL

STD: XII Material of the month – APRIL

Maths Theorem

Theorem 1.9: If A is non-singular square matrix of order n, then

i) $(\text{adj } A)^{-1} = (\text{adj } (A^{-1}) = \frac{1}{|A|} A$

ii) $|adj\ A| = |A|^{n-1}$

iii) $\text{adj } (\text{adj } A) = |A|^{n-2} A$

iv) $\text{adj } (\lambda A) = \lambda^{n-1} \text{adj}(A)$, λ is a non-zero sector

v) $|adj\ (adj\ A)| = |A|^{(n-1)^2}$

vi) $(\text{adj } A)^T = \text{adj } (A^T)$

Proof: Since a is non-singular square matrix, we have $|A| \neq 0$

i) $A^{-1} = \frac{1}{|A|} \text{adj } A \Rightarrow \text{adj } A = |A| A^{-1}$

Apply inverse on both sides, $(\text{adj } A)^{-1} = (|A|A^{-1})^{-1}$

$= (A^{-1})^{-1} . |A|^{-1}$

$= A . \frac{1}{|A|} \longrightarrow$ (a)

Replace A by A^{-1} in adj $A = |A| A^{-1}$

$\text{adj}(A^{-1}) = |A^{-1}| (A^{-1})^{-1}$

$\text{adj}(A^{-1}) = \frac{1}{|A|} . A \longrightarrow$ (b)

From (a) & (b), $(\text{adj } A)^{-1} = \text{adj}(A^{-1}) = \frac{1}{|A|} A$

ii) We know that, adj $A = A^{-1}|A|$

Apply determinant on both sides, $|adj\ A| = \left|A^{-1}|A|\right|$

$|\text{adj } A| = \left|A^{-1}|A|\right|$

$= |A|^n \ |A^{-1}|$

$= |A|^n . \frac{1}{|A|} = |A|^{n-1}$

$\therefore |\text{adj } A| = |A|^{n-1}$

iii) For any non-singular matrix of order n,

$B(\text{adj } B) = (\text{adj } B)B = |B| I_n$

Put B=adj A,

$\text{adj } A \ (\text{adj } (\text{adj } A)) = |\text{adj } A| I_n$

Pre-multiply by a on both sides,

$A(\text{adj } A(\text{adj } (\text{adj } A)) = A|A|^{n-1} I_n$ by (ii)

By associative property,

$A(\text{adj } A) \ (\text{adj } (\text{adj } A)) = |A|^{n-1} A$

$|A| \ I_n \ (\text{adj } (\text{adj } A)) = |A|^{n-1} A$

$|A| \ \text{adj } (\text{adj } A) = |A|^{n-1} A$

$\text{adj } (\text{adj } A) = \frac{|A|^{n-1} A}{|A|}$

$\text{adj } (\text{adj } A) = |A|^{n-2} . A$

iv) Replace A by λA in adj $A = |A| A^{-1}$,

we get, $\text{adj}(\lambda A) = |\lambda A| \ (\lambda A)^{-1}$

$= \lambda^n \ |A| \frac{1}{\lambda} A^{-1}$

$= \lambda^{n-1} \ |A| A^{-1}$

$= \lambda^{n-1}$ adj A

v) From (iii), $\text{adj}(\text{adj } A) = |A|^{n-2} . A$

Apply determinant on both sides,

$|\text{adj } (\text{adj } A)| = \left||A|^{n-2} . A\right|$

$= (|A|^{n-2})^n . |A|$

$= |A|^{n^2-2n} . |A|$

$= |A|^{n^2-2n+1}$

$= |A|^{(n-1)^2}$

vi) Since adj $A = |A| A^{-1}$

Replace A by A^T,

$\text{adj}(A^T) = |A^T| \ (A^T)^{-1}$

$= |A^T| \ (A^{-1})^T$

$= (A^{-1}|A|)^T$

$= (\text{adj } A)^T$

Theorem 1.10: If A and B are any two non-singular square matrices of order n, then adj(AB)=(adj B) (adj A)

Proof: Replacing A by AB in adj $A = |A| A^{-1}$

We get $\text{adj}(AB) = |AB| (AB^{-1})$

$= |A||B| \ B^{-1} \ A^{-1}$

$= (|B|B^{-1}) \ (|A|A^{-1})$

$\therefore \text{adj } (AB) = (\text{adj } B) \ (\text{adj } A)$